The reals modulo the integers

If x is a real number, we let $[x]$ denote the greatest integer less than or equal to x and call the quantity $[x]$ the **integer part** of x. The **fractional part** of x is then defined by $\langle x \rangle = x - [x]$. In particular, $\langle x \rangle \in [0, 1)$ for every $x \in \mathbb{R}$. For example, the integer and fractional parts of 2.7 are 2 and 0.7, respectively, while the integer and fractional parts of -3.4 are -4 and 0.6, respectively.

We may define a relation on $\mathbb{R}$ by saying that the two numbers x and y are equivalent, or congruent, if $x - y \in \mathbb{Z}$. We then write

$$x = y \bmod \mathbb{Z} \quad \text{or} \quad x = y \bmod 1.$$

This means that we identify two real numbers if they differ by an integer. Observe that any real number x is congruent to a unique number in $[0, 1)$ which is precisely $\langle x \rangle$, the fractional part of x. In effect, reducing a real number modulo $\mathbb{Z}$ means looking only at its fractional part and disregarding its integer part.

Now start with a real number $\gamma \neq 0$ and look at the sequence $\gamma, 2\gamma, 3\gamma, \ldots$. An intriguing question is to ask what happens to this sequence if we reduce it modulo $\mathbb{Z}$, that is, if we look at the sequence of fractional parts

$$\langle \gamma \rangle, \ \langle 2\gamma \rangle, \ \langle 3\gamma \rangle, \ \ldots.$$

Here are some simple observations:

(i) If γ is rational, then only finitely many numbers appearing in $\langle n\gamma \rangle$ are distinct.

(ii) If γ is irrational, then the numbers $\langle n\gamma \rangle$ are all distinct.

Indeed, for part (i), note that if $\gamma = p/q$, the first q terms in the sequence are

$$\langle p/q \rangle, \ \langle 2p/q \rangle, \ \ldots, \ \langle (q-1)p/q \rangle, \ \langle qp/q \rangle = 0.$$

The sequence then begins to repeat itself, since

$$\langle (q+1)p/q \rangle = \langle 1 + p/q \rangle = \langle p/q \rangle,$$

and so on. However, see Exercise 6 for a more refined result.

Also, for part (ii) assume that not all numbers are distinct. We therefore have $\langle n_1\gamma \rangle = \langle n_2\gamma \rangle$ for some $n_1 \neq n_2$; then $n_1\gamma - n_2\gamma \in \mathbb{Z}$, hence γ is rational, a contradiction.

In fact, it can be shown that if γ is irrational, then $\langle n\gamma \rangle$ is dense in the interval $[0, 1)$, a result originally proved by Kronecker. In other words, the sequence $\langle n\gamma \rangle$ hits every sub-interval of $[0, 1)$ (and hence it does so infinitely many times). We will obtain this fact as a corollary of a deeper theorem dealing with the uniform distribution of the sequence $\langle n\gamma \rangle$.

A sequence of numbers $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ in $[0, 1)$ is said to be **equidistributed** if for every interval $(a, b) \subset [0, 1)$,

$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : \xi_n \in (a, b)\}}{N} = b - a$$

where $\#A$ denotes the cardinality of the finite set A. This means that for large N, the proportion of numbers ξ_n in (a, b) with $n \leq N$ is equal to the ratio of the length of the interval (a, b) to the length of the interval $[0, 1)$. In other words, the sequence ξ_n sweeps out the whole interval evenly, and every sub-interval gets its fair share. Clearly, the ordering of the sequence is very important, as the next two examples illustrate.

EXAMPLE 1. The sequence

$$0, \ \frac{1}{2}, \ 0, \ \frac{1}{3}, \ \frac{2}{3}, \ 0, \ \frac{1}{4}, \ \frac{2}{4}, \ \frac{3}{4}, \ 0, \ \frac{1}{5}, \ \frac{2}{5}, \ \ldots$$

appears to be equidistributed since it passes over the interval $[0, 1)$ very evenly. Of course this is not a proof, and the reader is invited to give one. For a somewhat related example, see Exercise 8 with $\sigma = 1/2$.

EXAMPLE 2. Let $\{r_n\}_{n=1}^{\infty}$ be *any* enumeration of the rationals in $[0, 1)$. Then the sequence defined by

$$\xi_n = \begin{cases} r_{n/2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

is not equidistributed since "half" of the sequence is at 0. Nevertheless, this sequence is obviously dense.

We now arrive at the main theorem of this section.

Theorem 2.1 *If γ is irrational, then the sequence of fractional parts $\langle \gamma \rangle, \langle 2\gamma \rangle, \langle 3\gamma \rangle, \ldots$ is equidistributed in* $[0, 1)$.

In particular, $\langle n\gamma \rangle$ is dense in $[0, 1)$, and we get Kronecker's theorem as a corollary. In Figure 2 we illustrate the set of points $\langle \gamma \rangle, \langle 2\gamma \rangle, \langle 3\gamma \rangle, \ldots, \langle N\gamma \rangle$ for three different values of N when $\gamma = \sqrt{2}$.

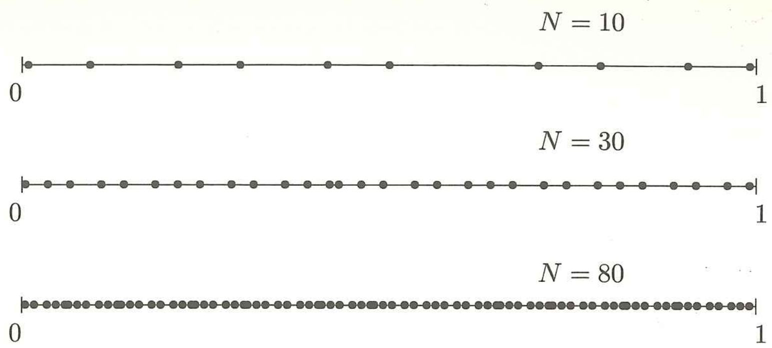

Figure 2. The sequence $\langle\gamma\rangle, \langle 2\gamma\rangle, \langle 3\gamma\rangle, \ldots, \langle N\gamma\rangle$ when $\gamma = \sqrt{2}$

Fix $(a, b) \subset [0, 1)$ and let $\chi_{(a,b)}(x)$ denote the characteristic function of the interval (a, b), that is, the function equal to 1 in (a, b) and 0 in $[0, 1) - (a, b)$. We may extend this function to $\mathbb{R}$ by periodicity (period 1), and still denote this extension by $\chi_{(a,b)}(x)$. Then, as a consequence of the definitions, we find that

$$\#\{1 \leq n \leq N : \langle n\gamma\rangle \in (a, b)\} = \sum_{n=1}^{N} \chi_{(a,b)}(n\gamma),$$

and the theorem can be reformulated as the statement that

$$\frac{1}{N}\sum_{n=1}^{N} \chi_{(a,b)}(n\gamma) \to \int_0^1 \chi_{(a,b)}(x)\,dx, \qquad \text{as } N \to \infty.$$

This step removes the difficulty of working with fractional parts and reduces the number theory to analysis.

The heart of the matter lies in the following result.

Lemma 2.2 *If f is continuous and periodic of period 1, and γ is irrational, then*

$$\frac{1}{N}\sum_{n=1}^{N} f(n\gamma) \to \int_0^1 f(x)\,dx \qquad \textit{as } N \to \infty.$$

The proof of the lemma is divided into three steps.

Step 1. We first check the validity of the limit in the case when f is one of the exponentials $1, e^{2\pi ix}, \ldots, e^{2\pi ikx}, \ldots$. If $f = 1$, the limit

surely holds. If $f = e^{2\pi ikx}$ with $k \neq 0$, then the integral is 0. Since γ is irrational, we have $e^{2\pi ik\gamma} \neq 1$, therefore

$$\frac{1}{N}\sum_{n=1}^{N} f(n\gamma) = \frac{e^{2\pi ik\gamma}}{N}\frac{1 - e^{2\pi ikN\gamma}}{1 - e^{2\pi ik\gamma}},$$

which goes to 0 as $N \to \infty$.

Step 2. It is clear that if f and g satisfy the lemma, then so does $Af + Bg$ for any $A, B \in \mathbb{C}$. Therefore, the first step implies that the lemma is true for all trigonometric polynomials.

Step 3. Let $\epsilon > 0$. If f is any continuous periodic function of period 1, choose a trigonometric polynomial P so that $\sup_{x\in\mathbb{R}} |f(x) - P(x)| < \epsilon/3$ (this is possible by Corollary 5.4 in Chapter 2). Then, by step 1, for all large N we have

$$\left|\frac{1}{N}\sum_{n=1}^{N} P(n\gamma) - \int_0^1 P(x)\,dx\right| < \epsilon/3.$$

Therefore

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N} f(n\gamma) - \int_0^1 f(x)\,dx\right| &\leq \frac{1}{N}\sum_{n=1}^{N} |f(n\gamma) - P(n\gamma)| + \\ &\quad + \left|\frac{1}{N}\sum_{n=1}^{N} P(n\gamma) - \int_0^1 P(x)\,dx\right| + \\ &\quad + \int_0^1 |P(x) - f(x)|\,dx \\ &< \epsilon,\end{aligned}$$

and the lemma is proved.

Now we can finish the proof of the theorem. Choose two continuous periodic functions f_ϵ^+ and f_ϵ^- of period 1 which approximate $\chi_{(a,b)}(x)$ on $[0, 1)$ from above and below; both f_ϵ^+ and f_ϵ^- are bounded by 1 and agree with $\chi_{(a,b)}(x)$ except in intervals of total length 2ϵ (see Figure 3). In particular, $f_\epsilon^-(x) \leq \chi_{(a,b)}(x) \leq f_\epsilon^+(x)$, and

$$b - a - 2\epsilon \leq \int_0^1 f_\epsilon^-(x)\,dx \quad \text{and} \quad \int_0^1 f_\epsilon^+(x)\,dx \leq b - a + 2\epsilon.$$

If $S_N = \frac{1}{N}\sum_{n=1}^{N} \chi_{(a,b)}(n\gamma)$, then we get

$$\frac{1}{N}\sum_{n=1}^{N} f_\epsilon^-(n\gamma) \leq S_N \leq \frac{1}{N}\sum_{n=1}^{N} f_\epsilon^+(n\gamma).$$

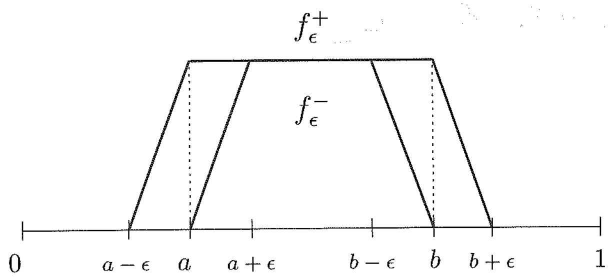

Figure 3. Approximations of $\chi_{(a,b)}(x)$

Therefore

$$b-a-2\epsilon \leq \liminf_{N\to\infty} S_N \quad \text{and} \quad \limsup_{N\to\infty} S_N \leq b-a+2\epsilon.$$

Since this is true for every $\epsilon > 0$, the limit $\lim_{N\to\infty} S_N$ exists and must equal $b-a$. This completes the proof of the equidistribution theorem.

This theorem has the following consequence.

Corollary 2.3 *The conclusion of Lemma 2.2 holds for every function f which is Riemann integrable in $[0,1]$, and periodic of period 1.*

Proof. Assume f is real-valued, and consider a partition of the interval $[0,1]$, say $0 = x_0 < x_1 < \cdots < x_N = 1$. Next, define $f_U(x) = \sup_{x_{j-1}\leq y\leq x_j} f(y)$ if $x \in [x_{j-1}, x_j)$ and $f_L(x) = \inf_{x_{j-1}\leq y\leq x_j} f(y)$ for $x \in (x_{j-1}, x_j)$. Then clearly $f_L \leq f \leq f_U$ and

$$\int_0^1 f_L(x)\,dx \leq \int_0^1 f(x)\,dx \leq \int_0^1 f_U(x)\,dx.$$

Moreover, by making the partition sufficiently fine we can guarantee that for a given $\epsilon > 0$,

$$\int_0^1 f_U(x)\,dx - \int_0^1 f_L(x)\,dx \leq \epsilon.$$

However,

$$\frac{1}{N}\sum_{n=1}^{N} f_L(n\gamma) \to \int_0^1 f_L(x)\,dx$$

by the theorem, because each f_L is a finite linear combination of characteristic functions of intervals; similarly we have

$$\frac{1}{N}\sum_{n=1}^{N} f_U(n\gamma) \to \int_0^1 f_U(x)\,dx.$$

From these two assertions we can conclude the proof of the corollary by using the previous approximation argument.

There is an interesting interpretation of the lemma and its corollary, in terms of a simple dynamical system. In this example, the underlying space is the circle parametrized by the angle θ. We also consider a mapping of this space to itself: here, we choose a rotation ρ of the circle by the angle $2\pi\gamma$, that is, the transformation $\rho : \theta \mapsto \theta + 2\pi\gamma$.

We want next to consider how this space, with its underlying action ρ, evolves in time. In other words, we wish to consider the iterates of ρ, namely ρ, ρ^2, ρ^3, ..., ρ^n where

$$\rho^n = \rho \circ \rho \circ \cdots \circ \rho : \theta \mapsto \theta + 2\pi n\gamma,$$

and where we think of the action ρ^n taking place at the time $t = n$.

To each Riemann integrable function f on the circle, we can also associate the corresponding effects of the rotation ρ, and obtain a sequence of functions

$$f(\theta),\ f(\rho(\theta)),\ f(\rho^2(\theta)),\ \ldots,\ f(\rho^n(\theta)),\ \ldots$$

with $f(\rho^n(\theta)) = f(\theta + 2\pi n\gamma)$. In this special context, the **ergodicity** of this system is then the statement that the "time average"

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\rho^n(\theta))$$

exists for each θ and equals the "space average"

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta,$$

whenever γ is irrational. In fact, this assertion is merely a rephrasing of Corollary 2.3, once we make the change of variables $\theta = 2\pi x$.

Returning to the problem of equidistributed sequences, we observe that the proof of Theorem 2.1 gives the following characterization.

Weyl's criterion. *A sequence of real numbers $\xi_1, \xi_2 \ldots$ in $[0,1)$ is equidistributed if and only if for all integers $k \neq 0$ one has*

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k \xi_n} \to 0, \quad \text{as } N \to \infty.$$

One direction of this theorem was in effect proved above, and the converse can be found in Exercise 7. In particular, we find that to understand the equidistributive properties of a sequence ξ_n, it suffices to estimate the size of the corresponding "exponential sum" $\sum_{n=1}^{N} e^{2\pi i k \xi_n}$. For example, it can be shown using Weyl's criterion that the sequence $\langle n^2\gamma\rangle$ is equidistributed whenever γ is irrational. This and other examples can be found in Exercises 8, and 9; also Problems 2, and 3.

As a last remark, we mention a nice geometric interpretation of the distribution properties of $\langle n\gamma\rangle$. Suppose that the sides of a square are reflecting mirrors and that a ray of light leaves a point inside the square. What kind of path will the light trace out?

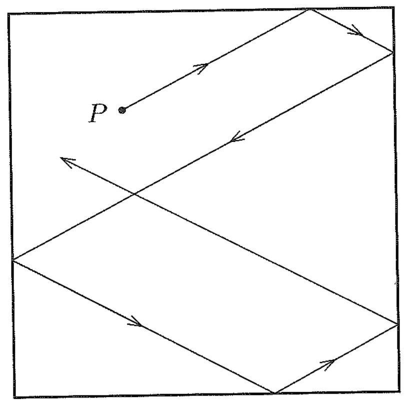

Figure 4. Reflection of a ray of light in a square

To solve this problem, the main idea is to consider the grid of the plane formed by successively reflecting the initial square across its sides. With an appropriate choice of axis, the path traced by the light in the square corresponds to the straight line $P + (t, \gamma t)$ in the plane. As a result, the reader may observe that the path will be either closed and periodic, or it will be dense in the square. The first of these situations

will happen if and only if the slope γ of the initial direction of the light (determined with respect to one of the sides of the square) is rational. In the second situation, when γ is irrational, the density follows from Kronecker's theorem. What stronger conclusion does one get from the equidistribution theorem?

3 A continuous but nowhere differentiable function

There are many obvious examples of continuous functions that are not differentiable at one point, say $f(x) = |x|$. It is almost as easy to construct a continuous function that is not differentiable at any given finite set of points, or even at appropriate sets containing countably many points. A more subtle problem is whether there exists a continuous function that is *nowhere* differentiable. In 1861, Riemann guessed that the function defined by

$$R(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2} \tag{5}$$

was nowhere differentiable. He was led to consider this function because of its close connection to the theta function which will be introduced in Chapter 5. Riemann never gave a proof, but mentioned this example in one of his lectures. This triggered the interest of Weierstrass who, in an attempt to find a proof, came across the first example of a continuous but nowhere differentiable function. Say $0 < b < 1$ and a is an integer > 1. In 1872 he proved that if $ab > 1 + 3\pi/2$, then the function

$$W(x) = \sum_{n=1}^{\infty} b^n \cos(a^n x)$$

is nowhere differentiable.

But the story is not complete without a final word about Riemann's original function. In 1916 Hardy showed that R is not differentiable at all irrational multiples of π, and also at certain rational multiples of π. However, it was not until much later, in 1969, that Gerver completely settled the problem, first by proving that the function R *is* actually differentiable at all the rational multiples of π of the form $\pi p/q$ with p and q odd integers, and then by showing that R is not differentiable in all of the remaining cases.

In this section, we prove the following theorem.